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# EQUIVARIANT LOCAL INDEX AND SYMPLECTIC CUT (New transformation groups and its related topics)

AUTHOR(S):

Yoshida, Takahiko

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# EQUIVARIANT LOCAL INDEX AND SYMPLECTIC CUT

TAKAHIKO YOSHIDA

## 1. INTRODUCTION

This is a survey article of [7]. In the joint work [1, 2, 3] with Fujita and Furuta we developed an index theory for Dirac-type operators on possibly non-compact Riemannian manifolds. We call the index in our theory the *local index* and also call its equivariant version the *equivariant local index*. As an application to Hamiltonian  $S^1$ -actions on prequantizable closed symplectic manifolds we can show that the equivariant Riemann-Roch index is obtained as the sum of the equivariant local indices for the inverse images of the integer lattice points by the moment map. When the lattice point is a regular value of the moment map we can compute its equivariant local index, see [3, 6]. So the problem is how to compute the equivariant local index when the lattice point is a critical value. The purpose of this paper is to give a formula for the equivariant local index for the reduced space in a symplectic cut space which is a special case of critical lattice points (Theorem 3.4).

This paper is organized as follows. In Section 2 we briefly recall the equivariant local index in the case of the Hamiltonian  $S^1$ -actions. After that we give the setting and the main theorem in Section 3.

**Notation.** In this paper we use the notation  $\mathbb{C}_{(n)}$  for the irreducible representation of  $S^1$  with weight  $n$ .

## 2. EQUIVARIANT LOCAL INDEX

Let  $(M, \omega)$  be a prequantizable Hamiltonian  $S^1$ -manifold and  $(L, \nabla^L)$  an  $S^1$ -equivariant prequantum line bundle on  $(M, \omega)$ . We do not assume  $M$  is compact. Since all orbits are isotropic the restriction of  $(L, \nabla^L)$  to each orbit is flat.

**Definition 2.1.** An orbit  $\mathcal{O}$  is said to be *L-acyclic* if  $H^0(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}) = 0$ .

Let  $V$  be an  $S^1$ -invariant open set whose complement is compact and which contains only *L-acyclic* orbits. For these data we give the following theorem.

**Theorem 2.2** ([1, 2, 3]). *There exists an element  $\text{ind}_{S^1}(M, V; L) \in R(S^1)$  of the representation ring such that  $\text{ind}_{S^1}(M, V; L)$  satisfies the following properties:*

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- (1)  $\text{ind}_{S^1}(M, V; L)$  is invariant under continuous deformation of the data.
- (2) If  $M$  is closed, then  $\text{ind}_{S^1}(M, V; L)$  is equal to the equivariant Riemann-Roch index  $\text{ind}_{S^1}(M; L)$ .
- (3) If  $M'$  is an  $S^1$ -invariant open neighborhood of  $M \setminus V$ , then  $\text{ind}_{S^1}(M, V; L)$  satisfies the following excision property

$$\text{ind}_{S^1}(M, V; L) = \text{ind}_{S^1}(M', M' \cap V; L|_{M'}).$$

- (4)  $\text{ind}_{S^1}(M, V; L)$  satisfies a product formula.

We call  $\text{ind}_{S^1}(M, V; L)$  the *equivariant local index*.

**Example 2.3.** For small positive real number  $\varepsilon > 0$  which is less than 1 let  $D_\varepsilon(\mathbb{C}_{(1)}) = \{z \in \mathbb{C}_{(1)} \mid |z| < \varepsilon\}$  be the 2-dimensional disc of radius  $\varepsilon$ . As  $(L, \nabla^L) \rightarrow (M, \omega)$  we consider

$$\left( D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(m)}, d + \frac{1}{2}(zd\bar{z} - \bar{z}dz) \right) \rightarrow \left( D_\varepsilon(\mathbb{C}_{(1)}), \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} \right).$$

First let us detect non  $L$ -acyclic orbits. Suppose the orbit  $\mathcal{O}$  through  $z \in D_\varepsilon(\mathbb{C}_{(1)})$  has a non-trivial parallel section  $s \in H^0(\mathcal{O}; (L, \nabla)|_{\mathcal{O}})$ . Then  $s$  satisfies the following equation

$$0 = \nabla_{\partial_\theta}^L s = \frac{\partial s}{\partial \theta} - 2\pi\sqrt{-1}r^2 s,$$

where we use the polar coordinates  $z = re^{2\pi\sqrt{-1}\theta}$ . Hence  $s$  is of the form  $s(\theta) = s_0 e^{2\pi\sqrt{-1}r^2\theta}$  for some non-zero constant  $s_0$ . Since  $s$  is a global section on  $\mathcal{O}$   $s$  satisfies  $s(0) = s(1)$ . This implies  $r = 0$ .

Next, we put  $V = D_\varepsilon(\mathbb{C}_{(1)}) \setminus \{0\}$  and let us compute  $\text{ind}_{S^1}(M, V; L)$ . We recall the definition of  $\text{ind}_{S^1}(N, V; L)$ . For  $t \geq 0$  consider the following perturbation of the  $\text{Spin}^c$  Dirac operator  $D: \Gamma(\wedge^{0,*}T^*M \otimes L) \rightarrow \Gamma(\wedge^{0,*}T^*M \otimes L)$  associated with the standard Hermitian structure on  $M = D_\varepsilon(\mathbb{C}_{(1)})$

$$D_t = D + t\rho D_{S^1},$$

where  $\rho$  is a cut-off function of  $V$  and  $D_{S^1}$  is a first order formally self-adjoint differential operator of degree-one

$$D_{S^1}: \Gamma((\wedge^*T^*M^{0,1} \otimes L)|_V) \rightarrow \Gamma((\wedge^*T^*M^{0,1} \otimes L)|_V)$$

that satisfies the following conditions:

- (1)  $D_{S^1}$  contains only derivatives along orbits.
- (2) The restriction  $D_{S^1}|_{\mathcal{O}}$  to an orbit  $\mathcal{O}$  is the de Rham operator with coefficients in  $L|_{\mathcal{O}}$ .
- (3) For any  $S^1$ -equivariant section  $u$  of the normal bundle  $\nu_{\mathcal{O}}$  of  $\mathcal{O}$  in  $M$ ,  $D_{S^1}$  anti-commutes with the Clifford multiplication of  $u$ .

See [1, 2, 3] for more details. From the second condition and  $\{0\}$  is the unique non  $L$ -acyclic orbit we can see  $\ker(D_{S^1}|_{\mathcal{O}}) = 0$  for all orbits  $\mathcal{O} \neq \{0\}$ . Extend the complement of a neighborhood of 0 in  $D_\varepsilon(\mathbb{C}_{(1)})$  cylindrically so that all the data are translationally invariant. Then we showed in [1, 2] that for a sufficiently large  $t$   $D_t$  is Fredholm, namely,  $\ker D_t \cap L^2$  is finite

dimensional and its super-dimension is independent of a sufficiently large  $t$ . So we define

$$\text{ind}_{S^1}(M, V; L) = \ker D_t^0 \cap L^2 - \ker D_t^1 \cap L^2$$

for a sufficiently large  $t$ . In this case, by the direct computation using the Fourier expansion of  $s$  with respect to  $\theta$ , we can show that

$$\ker D_t^0 \cap L^2 \cong \mathbb{C}, \quad \ker D_t^1 \cap L^2 = 0,$$

and  $\ker D_t^0 \cap L^2$  is spanned by a certain  $L^2$ -function  $a_0(r)$  on  $D_\varepsilon(\mathbb{C}_{(1)})$  which depends only on  $r = |z|$ . Since the  $S^1$ -action on  $\ker D_t^0 \cap L^2$  is given by pull-back and the  $S^1$ -action on the fiber is given by  $\mathbb{C}_{(m)}$  we obtain

$$\begin{aligned} \text{ind}_{S^1}(M, V; L) &= \text{ind}_{S^1}(D_\varepsilon(\mathbb{C}_{(1)}), D_\varepsilon(\mathbb{C}_{(1)}) \setminus \{0\}; D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(m)}) \\ &= \mathbb{C}_{(-m)}. \end{aligned}$$

For more details see [1, Remark 6.10], or [5, Section 5.3.2].

It is well-known that the lift of  $S^1$ -action on  $M$  to  $L$  defines the moment map  $\mu: M \rightarrow \mathbb{R}$  by the Kostant formula

$$(2.1) \quad \mathcal{L}_X s = \nabla_X^L s + 2\pi\sqrt{-1}\mu s,$$

where  $s$  is a section of  $L$ ,  $X$  is the vector field which generates the  $S^1$ -action on  $(M, \omega)$ , and  $\mathcal{L}_X s$  is the Lie derivative which is defined by

$$\mathcal{L}_X s(x) = \frac{d}{d\theta} \Big|_{\theta=0} e^{-2\pi\sqrt{-1}\theta} s(e^{2\pi\sqrt{-1}\theta} x).$$

**Lemma 2.4.** *If an orbit  $\mathcal{O}$  is not  $L$ -acyclic, namely,  $H^0(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}) \neq 0$ , then,  $\mu(\mathcal{O}) \in \mathbb{Z}$ .*

If  $M$  is closed, then we have the following localization formula for the equivariant Riemann-Roch index.

**Corollary 2.5.** *Suppose  $M$  is closed. For  $i \in \mu(M) \cap \mathbb{Z}$  let  $V_i$  be an  $S^1$ -invariant open neighborhood of  $\mu^{-1}(i)$  such that they are mutually disjoint, namely,  $V_i \cap V_j \neq \emptyset$  for all  $i \neq j$ . Then,*

$$(2.2) \quad \text{ind}_{S^1}(M; L) = \bigoplus_{i \in \mu(M) \cap \mathbb{Z}} \text{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i}).$$

### 3. THE SETTING AND THE MAIN THEOREM

Let  $(M, \omega)$  be a Hamiltonian  $S^1$ -space with moment map  $\mu: M \rightarrow \mathbb{R}$ . For a real number  $n$  the cut space  $\overline{M}_{\mu \leq n}$  of  $(M, \omega)$  by the symplectic cutting [4] is the reduced space of the diagonal  $S^1$ -action on  $(M, \omega) \times \left(\mathbb{C}_{(1)}, \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}\right)$ , namely,

$$\overline{M}_{\mu \leq n} = \left\{ (x, z) \in (M, \omega) \times \left(\mathbb{C}_{(1)}, \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}\right) \mid \mu(x) + |z|^2 = n \right\} / S^1.$$

We denote the reduced space  $\mu^{-1}(n)/S^1$  by  $M_n$ .

**Proposition 3.1.** (1) If  $S^1$  acts on  $\mu^{-1}(n)$  freely, then,  $\overline{M}_{\mu \leq n}$  is a smooth Hamiltonian  $S^1$ -space. The  $S^1$ -action is given as

$$(3.1) \quad t[x, z] = [tx, z]$$

for  $t \in S^1$  and  $[x, z] \in \overline{M}_{\mu \leq n}$ .

(2) Under the assumption in (1), the reduced space  $M_n$  and  $\{x \in M \mid \mu(x) \leq n\}$  are symplectically embedded into  $\overline{M}_{\mu \leq n}$  by  $M_n \ni [x] \mapsto [x, 0] \in \overline{M}_{\mu \leq n}$  and  $\{x \in M \mid \mu(x) \leq n\} \ni x \mapsto [x, \sqrt{n - \mu(x)}] \in \overline{M}_{\mu \leq n}$ , respectively. In particular,  $\overline{M}_{\mu \leq n}$  can be identified with the disjoint union  $\{x \in M \mid \mu(x) \leq n\} \amalg M_n$  and with this identification  $M_n$  is fixed by the  $S^1$ -action (3.1).

Suppose that  $(M, \omega)$  is equipped with a prequantum line bundle  $(L, \nabla^L) \rightarrow (M, \omega)$  and the  $S^1$ -action lifts to  $(L, \nabla^L)$  in such a way that  $\mu$  satisfies the Kostant formula (2.1).

**Proposition 3.2.** If  $n$  is an integer and the  $S^1$ -action on  $\mu^{-1}(n)$  is free, then  $\overline{M}_{\mu \leq n}$  is prequantizable. In this case a prequantum line bundle  $(\overline{L}, \nabla^{\overline{L}})$  on  $\overline{M}_{\mu \leq n}$  is given by

$$(\overline{L}, \nabla^{\overline{L}}) = ((L, \nabla^L) \otimes \mathbb{C}_{(n)}) \boxtimes \left( \mathbb{C}_{(1)} \times \mathbb{C}_{(0)}, d + \frac{1}{2}(zd\bar{z} - \bar{z}dz) \right) \Big|_{\Phi^{-1}(0)} / S^1,$$

where  $\Phi$  is the moment map  $\Phi: M \times \mathbb{C}_{(1)} \rightarrow \mathbb{R}$  associated to the lift of the diagonal  $S^1$ -action which is written as  $\Phi(x, z) = \mu(x) + |z|^2 - n$ , and the lift of the  $S^1$ -action (3.1) on  $\overline{M}_{\mu \leq n}$  to  $(\overline{L}, \nabla^{\overline{L}})$  is given by

$$(3.2) \quad t[u \otimes v \boxtimes (z, w)] = [(tu) \otimes v \boxtimes (z, w)]$$

for  $t \in S^1$  and  $[u \otimes v \boxtimes (z, w)] \in \overline{L}$ . The moment map  $\overline{\mu}: \overline{M}_{\mu \leq n} \rightarrow \mathbb{R}$  associated with the lift (3.2) is written as  $\overline{\mu}([x, z]) = \mu(x) = n - |z|^2$ .

See [4] for more details.

**Remark 3.3.** We denote the restriction of  $(\overline{L}, \nabla^{\overline{L}})$  to  $M_n$  by  $(L_n, \nabla^{L_n})$ .  $(L_n, \nabla^{L_n})$  is a prequantum line bundle on  $M_n$ . The  $S^1$ -action (3.2) on  $L_n$  is given by the fiberwise multiplication with weight  $n$ . Recall that  $M_n$  is fixed by the  $S^1$ -action (3.1). See Proposition 3.1.

Suppose that  $n$  be an integer and the  $S^1$ -action on  $\mu^{-1}(n)$  is free. Then, the cut space  $\overline{M}_{\mu \leq n}$  becomes a prequantizable Hamiltonian  $S^1$ -manifold and the  $S^1$ -equivariant prequantum line bundle  $(\overline{L}, \nabla^{\overline{L}})$  is given by Proposition 3.1 and Proposition 3.2.

Suppose also that  $\mu^{-1}(n)$  is compact. We take a sufficiently small  $S^1$ -invariant open neighborhood  $O$  of  $M_n$  in  $\overline{M}_{\mu \leq n}$  so that the intersection  $\overline{\mu}(O) \cap \mathbb{Z}$  consists of the unique point  $n$ . Then we can define the equivariant local index  $\text{ind}_{S^1}(O, O \setminus M_n; \overline{L}|_O)$  of  $M_n$  in  $\overline{M}_{\mu \leq n}$ . We give the following formula for  $\text{ind}_{S^1}(O, O \setminus M_n; \overline{L}|_O)$ .

**Theorem 3.4.** Let  $(M, \omega)$ ,  $(L, \nabla^L)$ , and  $\mu$  be as above. Let  $n$  be an integer. Suppose  $S^1$  acts on  $\mu^{-1}(n)$  freely and  $\mu^{-1}(n)$  is compact. Let  $O$  be a sufficiently small  $S^1$ -invariant open neighborhood of  $M_n$  in  $\overline{M}_{\mu \leq n}$  which satisfies  $\overline{\mu}(O) \cap \mathbb{Z} = \{n\}$ . Then, the equivariant local index is given as

$$\text{ind}_{S^1}(O, O \setminus M_n; \overline{L}|_O) = \text{ind}(M_n; L_n) \mathbb{C}_{(n)},$$

where  $\text{ind}(M_n; L_n)$  is the Riemann-Roch number of  $M_n$ .

**Remark 3.5.** By replacing  $\mathbb{C}_{(1)}$  with  $\mathbb{C}_{(-1)}$  in the above construction we obtain the other cut space  $\overline{M}_{\mu \geq n} = \{(x, z) \in M \times \mathbb{C}_{-1} : \mu(x) - |z|^2 = n\} / S^1$ . Theorem 3.4 also holds for  $\overline{M}_{\mu \geq n}$ .

*The outline of the proof of Theorem 3.4.* By the definition of the symplectic cutting, the normal bundle  $\nu$  of  $M_n$  in  $\overline{M}_{\mu \leq n}$  is given by

$$\nu = \mu^{-1}(N) \times_{S^1} \mathbb{C}_{(1)}.$$

For a sufficiently small  $\varepsilon > 0$  let  $D_\varepsilon(\mathbb{C}_{(1)}) = \{z \in \mathbb{C}_{(1)} : |z| < \varepsilon\}$  be the open disc of radius  $\varepsilon$ . We put  $D_\varepsilon(\nu) = \mu^{-1}(n) \times_{S^1} D_\varepsilon(\mathbb{C}_{(1)})$ , and define an  $S^1$ -action on  $D_\varepsilon(\nu)$  by

$$(3.3) \quad t[x, z] = [tx, z].$$

Let  $p: D_\varepsilon(\nu) \rightarrow M_n$  be the natural projection. We define a complex line bundle  $L_{D_\varepsilon(\nu)}$  on  $D_\varepsilon(\nu)$  by

$$L_{D_\varepsilon(\nu)} = p^* L_n \otimes (\mu^{-1}(n) \times_{S^1} (D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(0)})),$$

and define an lift of the  $S^1$ -action (3.3) to  $L_{D_\varepsilon(\nu)}$  by

$$(3.4) \quad t([x, z], [u \otimes v]) \otimes [x', z', w] = ([tx, z], [(tu) \otimes v]) \otimes [tx', z', w].$$

Then we can show that for a sufficiently small  $\varepsilon > 0$   $L_{D_\varepsilon(\nu)}$  on  $D_\varepsilon(\nu)$  is equivariantly identified with  $\overline{L}$  restricted to certain neighborhood of  $M_n$  in  $\overline{M}_{\mu \leq n}$ . By using this identification and the equivariant version of the product formula [2, Theorem 8.8] we obtain

$$\begin{aligned} & \text{ind}_{S^1}(O, O \setminus M_n; \overline{L}|_O) \\ &= \text{ind}_{S^1}(D_\varepsilon(\nu), D_\varepsilon(\nu) \setminus M_n; L_{D_\varepsilon(\nu)}) \\ &= \text{ind}_{S^1}(M_n; L_n \otimes \mu^{-1}(n) \times_{S^1} \text{ind}_{S^1}(D_\varepsilon(\mathbb{C}_{(1)}), D_\varepsilon(\mathbb{C}_{(1)}) \setminus \{0\}; D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(0)})). \end{aligned}$$

Note that the product formula for the  $S^1$ -equivariant local index holds since the  $S^1$ -action preserves all the data. See [3, Section 6.2] for more details. From Example 2.3 the equivariant local index  $\text{ind}_{S^1}(D_\varepsilon(\mathbb{C}_{(1)}), D_\varepsilon(\mathbb{C}_{(1)}) \setminus \{0\}; D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(0)})$  is equal to  $\mathbb{C}_{(0)}$ . By definition,  $L_n$  is naturally identified with the restriction of  $\overline{L}$  to  $M_n$ . With this identification we can see that the restriction of the  $S^1$ -action (3.2) to  $L_n \rightarrow M_n$  is nothing but the fiberwise multiplication of  $t^{-n}$ . Since the  $S^1$ -action on  $\text{ind}_{S^1}(M_n; L_n)$  is defined by the pull-back, the  $S^1$ -action on  $\text{ind}_{S^1}(M_n; L_n)$  is given by the multiplication of  $t^n$  as we mentioned in Remark 3.3. This proves the theorem.  $\square$

**Example 3.6** (Complex projective space). As  $(L, \nabla) \rightarrow (M, \omega)$  we adopt

$$\left( (\mathbb{C}_{(1)})^m \times \mathbb{C}_{(0)}, d + \frac{1}{2} \sum_{i=1}^m (z_i d\bar{z}_i - \bar{z}_i dz_i) \right) \rightarrow \left( (\mathbb{C}_{(1)})^m, \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^m dz_i \wedge d\bar{z}_i \right).$$

For  $n = 1$  the obtained  $\overline{M}_{\mu \leq n}$ ,  $\overline{L}$ , and  $M_n$  are  $\mathbb{C}P^m$ ,  $\mathcal{O}(1)$ , and  $\mathbb{C}P^{m-1}$ , respectively. The induced  $S^1$ -actions on  $\mathbb{C}P^m$  and  $\mathcal{O}(1)$  are given by

$$(3.5) \quad \begin{aligned} t[z_0 : z_1 : \cdots : z_m] &= [z_0 : tz_1 : \cdots : tz_m], \\ t[z_0 : z_1 : \cdots : z_m, w] &= [z_0 : tz_1 : \cdots : tz_m, w]. \end{aligned}$$

The moment map  $\bar{\mu}$  associated to the  $S^1$ -action (3.5) is given by  $\bar{\mu}([z_0 : \cdots : z_m]) = \sum_{i=1}^m |z_i|^2$ . For  $k = 0, 1$  let  $O_k$  be a sufficient small  $S^1$ -invariant open neighborhood of  $\bar{\mu}^{-1}(k)$ . Then the equivariant local index  $\text{ind}_{S^1}(O_k, O_k \setminus \bar{\mu}^{-1}(k); \bar{L}|_{O_k})$  is defined and By Corollary 2.2 the equivariant Riemann-Roch index  $\text{ind}_{S^1}(\bar{M}_{\mu \leq n}, \bar{L})$  satisfies following equality

$$(3.6) \quad \begin{aligned} \text{ind}_{S^1}(\bar{M}_{\mu \leq n}, \bar{L}) &= \text{ind}_{S^1}(O_0, O_0 \setminus \bar{\mu}^{-1}(0); \bar{L}|_{O_0}) \\ &\quad + \text{ind}_{S^1}(O_1, O_1 \setminus \bar{\mu}^{-1}(1); \bar{L}|_{O_1}). \end{aligned}$$

The left hand side is computed as

$$(3.7) \quad \text{ind}_{S^1}(\bar{M}_{\mu \leq n}, \bar{L}) = \text{ind}_{S^1}(\mathbb{C}P^m, \mathcal{O}(1)) = \mathbb{C}_{(0)} \oplus m\mathbb{C}_{(1)}.$$

For  $k = 1$ , since  $\bar{\mu}^{-1}(1) = M_n$ , by Theorem 3.4  $\text{ind}_{S^1}(O_1, O_1 \setminus \bar{\mu}^{-1}(1); \bar{L}|_{O_1})$  is given as

$$(3.8) \quad \begin{aligned} \text{ind}_{S^1}(O_1, O_1 \setminus \bar{\mu}^{-1}(1); \bar{L}|_{O_1}) &= \text{ind}_{S^1}(O_1, O_1 \setminus M_n; \bar{L}|_{O_1}) \\ &= \text{ind}(\mathbb{C}P^{m-1}; \mathcal{O}(1))\mathbb{C}_{(1)} \\ &= m\mathbb{C}_{(1)}. \end{aligned}$$

For  $k = 0$ , it is easy to see that  $\bar{\mu}^{-1}(0) = \{[z_0 : 0 : \cdots : 0]\}$  and  $(\bar{L}, \nabla^{\bar{L}})|_{[z_0:0:\cdots:0]} \cong (\mathbb{C}_{(0)}, d + \frac{1}{2}(\bar{z}dz - z d\bar{z}))$ . We can take  $O_0$  in such a way that  $O_0$  is identified with a sufficiently small open disc  $D = \{(z_1, \dots, z_m) \in \mathbb{C}^m : \sum_{i=1}^m |z_i|^2 \leq \varepsilon\}$  with  $S^1$ -action  $t(z_1, \dots, z_m) = (tz_1, \dots, tz_m)$ . Then, by (3.6), (3.7), and (3.8) we obtain the following formula

$$\text{ind}_{S^1}(D, D \setminus \{0\}; D \times \mathbb{C}_0) = \mathbb{C}_{(0)}.$$

In the case of  $m = 1$  this formula can be obtained in [1, Remark 6.10] and [5, Section 5.3.2].

**Example 3.7** (Exceptional divisor). Let  $n$  and  $(L, \nabla) \rightarrow (M, \omega)$  be as in Example 3.6. Then the obtained cut space  $\bar{M}_{\mu \geq n}$  is the blow-up  $\tilde{\mathbb{C}}^m$  of the origin in  $\mathbb{C}^m$ , and  $M_n$  and  $L_n$  are the exceptional divisor  $\mathbb{C}P^{m-1}$  and  $\mathcal{O}(n)$ , respectively. We take a sufficiently small invariant open neighborhood  $O$  of  $M_n$ . Then, by Theorem 3.4 the equivariant local index  $\text{ind}_{S^1}(O, O \setminus M_n; \bar{L}|_O)$  is given by

$$\text{ind}_{S^1}(O, O \setminus M_n; \bar{L}|_O) = \text{ind}(\mathbb{C}P^{m-1}; \mathcal{O}(n))\mathbb{C}_{(n)} = \binom{m-1+n}{m-1}\mathbb{C}_{(n)}.$$

## REFERENCES

1. H. Fujita, M. Furuta, and T. Yoshida, *Torus fibrations and localization of index I*, J. Math. Sci. Univ. Tokyo **17** (2010), no. 1, 1–26.
2. ———, *Torus fibrations and localization of index II*, Comm. Math. Phys. **326** (2014), no. 3, 585–633.
3. ———, *Torus fibrations and localization of index III*, Comm. Math. Phys. **327** (2014), no. 3, 665–689.
4. E. Lerman, *Symplectic cuts*, Math. Res. Lett. **2** (1995), no. 3, 247–258.
5. T. Yoshida, *RR = #BS via localization of index*, Trends in Math. **12** (2010), no. 1, 1–41.
6. ———, *Equivariant local index*, RIMS Kôkyûroku Bessatsu **B39** (2013), 215–232.
7. ———, *The equivariant local index of the reduced space in the symplectic cutting*, arXiv:1402.6437, 2014.

DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE AND TECHNOLOGY, MEIJI UNIVERSITY, 1-1-1 HIGASHIMITA, TAMA-KU, KAWASAKI, 214-8571, JAPAN

*E-mail address:* `takahiko@meiji.ac.jp`